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Dissipative Operators Generated by the Sturm–Liouville Differential Expression in the Weyl Limit Circle Case

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In this paper, using Livšic's theorem, we investigate the problem of completeness of the system of eigenfunctions and associated functions of dissipative operators generated by the Sturm–Liouville differential expression on the semi-axis in Weyl's limit-circle case. © 2001 Academic Press

1. INTRODUCTION

Let us consider the differential expression

$$\ell(y) = -(p(x)y')' + q(x)y, \qquad x \in \mathbf{R}_+ = [0, \infty),$$

where p, q are real-valued functions on $\mathbf{R}_+$ and $p^{-1}, q \in L_{1,\mathrm{loc}}(\mathbf{R}_+)$. We denote by $L_{\min}$ the minimal symmetric operator in $L_2(\mathbf{R}_+)$ generated by $\ell(y)$. The operator $L_{\min}$ has defect index $(1, 1)$ or $(2, 2)$. In this paper, we always assume that the functions p and q are such that the operator $L_{\min}$ has the defect index $(2, 2)$; in other words, Weyl's limit circle case holds for the differential expressions $\ell(y)$ [1, p. 225]). We can give several sufficient

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conditions in which the operator $L_{\min}$ has the defect index $(2,2)$. Such a condition is given in [7, p. 196].

Assume that $p, q, q' \in AC_{\mathrm{loc}}(\mathbf{R}_+)$, $p'p^{-1} \in L_1(\mathbf{R}_+)$, and that q' and q'' have a constant sign in $[x_0, \infty)$ for sufficiently large x_0. Let $q \to -\infty$ and $q' = O(|q|^{\beta})$ as $x \to \infty$, where $0 < \beta < \frac{3}{2}$. If

$$\int^{\infty} |q(x)|^{-1/2}\, dx < \infty,$$

then the operator $L_{\min}$ has the defect index $(2,2)$.

Let

$$\Omega = \{y \in L_2(\mathbf{R}_+) : y, py' \in AC_{\mathrm{loc}}(\mathbf{R}_+), l(y) \in L_2(\mathbf{R}_+)\}.$$

Note that Ω is the domain of the maximal operator $L_{\max}$ generated by $\ell(y)$ and $L_{\max} = L^*_{\min}$.

For all $y_1, y_2 \in \Omega$ we set

$$[y_1, y_2]_x = y_1(x)\big(p(x)\overline{y_2'(x)}\big) - (p(x)y_1'(x))\overline{y_2(x)},$$

where the bar over a function denotes the complex conjugate.

Green's formula

$$\int_0^b \ell(y_1)\overline{y_2}\, dx - \int_0^b y_1 \ell(\overline{y_2})\, dx = [y_1, y_2]_b - [y_1, y_2]_0 \tag{1.1}$$

implies that, for all functions $y_1, y_2 \in \Omega$, the limit $[y_1, y_2]_\infty = \lim_{x\to\infty}[y_1, y_2]_x$ exists and is finite.

Let $W_x[y_1, y_2]$ denote the Wronskian of two solutions $y_1 = y_1(x, \lambda)$ and $y_2 = y_2(x, \lambda)$ of the equation

$$-(p(x)y')' + q(x)y = \lambda y, \qquad x \in \mathbf{R}_+, \tag{1.2}$$

where λ is a complex parameter. Then $W_x[y_1, y_2] = [y_1, \overline{y_2}]_x$.

Denote by $\theta(x, \lambda)$ and $\varphi(x, \lambda)$ the solution of (1.2) satisfying the initial conditions

$$\theta(0, \lambda) = \cos\alpha, \qquad p(0)\theta'(0, \lambda) = \sin\alpha,$$
$$\varphi(0, \lambda) = -\sin\alpha, \qquad p(0)\varphi'(0, \lambda) = \cos\alpha,$$

where α is a some real number. The solutions $\theta(x, \lambda)$ and $\varphi(x, \lambda)$ form a fundamental system of solutions of (1.2) and they are entire functions of λ. Also, they are real functions for real values of λ. Since the operator $L_{\min}$ has the defect index $(2,2)$, the solutions $\theta(x, \lambda)$ and $\varphi(x, \lambda)$ belong to

$L_2(\mathbf{R}_+)$. Let $z(x) = \theta(x, 0)$ and $u(x) = \varphi(x, 0)$. So $z(x)$ and $u(x)$ are solutions of the equation $\ell(y) = 0$, satisfying the initial conditions.

$$z(0) = \cos\alpha, \qquad p(0)z'(0) = \sin\alpha,$$
$$u(0) = -\sin\alpha, \qquad p(0)u'(0) = \cos\alpha.$$

We have $z, u \in L_2(\mathbf{R}_+)$; moreover $z, u \in \Omega$. Consequently for each $y \in \Omega$ the values $[y, z]_\infty$ and $[y, u]_\infty$ exist and are finite.

Let $D(L)$ denote the set of all functions $y \in \Omega$ satisfying the boundary conditions

$$y(0)\cos\alpha + p(0)y'(0)\sin\alpha = 0, \qquad [y, z]_\infty - h[y, u]_\infty = 0, \quad (1.3)$$

where h is a some complex number [4, 5]. In $L_2(\mathbf{R}_+)$ we define the operator L with the domain $D(L)$ and

$$Ly = \ell(y),$$

for all $y \in D(L)$. We also denote by M the operator generated, in $L_2(\mathbf{R}_+)$, by the differential expression $\ell(y)$ and the boundary conditions

$$p(0)y'(0) - h_0 y_0 = 0, \qquad [y, z]_\infty \cos\alpha + [y, u]_\infty \sin\alpha = 0,$$

where h_0 is a some complex number. In the following we will assume that $\operatorname{Im} h > 0$ and $\operatorname{Im} h_0 > 0$. It is clear that the operators L and M are non-selfadjoint.

One general method of the spectral analysis of non-selfadjoint differential operators is the method of contour integration of the resolvent. It is connected with a finite estimate of the resolvent on expanding contours separating the spectrum. Since the resolvents of the operators L and M have no fine estimate, this method cannot be applied to them.

In this paper, using Livšic's theorem and the determinant of perturbations connected with the operators L and M, we prove that the system of all eigenvectors and associated vectors of this operators are complete in $L_2(\mathbf{R}_+)$.

2. PRELIMINARIES

Let A denote the linear non-selfadjoint operator in the Hilbert space H with the domain $D(A)$. The element $y \in D(A)$, $y \neq 0$ is called a root vector of the operator A corresponding to the eigenvalue λ_0, if all powers of A are defined on this element and $(A - \lambda_0 I)^n y = 0$ for some integer $n > 0$. The set of all root vectors of A corresponding to the same eigenvalue λ_0 with the vector $y = 0$ forms a linear set N_{λ_0} and is called the root lineal. The dimension of the lineal N_{λ_0} is called the algebraic multiplicity of the eigenvalue λ_0. The root lineal N_{λ_0} coincides with the linear span of all eigenvectors and associated vectors of A corresponding to the eigenvalue λ_0. Consequently the completeness of the system of all eigenvectors and associated vectors of A is equivalent to the completeness of the system of all root vectors of this operator.

If $\operatorname{Im}(Ay, y) \geq 0$ for all $y \in D(A)$, then the operator A is called dissipative. For a bounded operator A (defined on all of H) the condition of dissipativeness is equivalent to the condition that its imaginary component $A_J = \frac{1}{2i}(A - A^*)$ is nonnegative.

We will denote the class of all nuclear and Hilbert–Schmidt operators in H by σ_1 and σ_2, respectively. Let $\{\mu_j(A)\}_{j=1}^{v(A)}$ be a sequence of all nonzero eigenvalues of $A \in \sigma_p$, $p = 1, 2$, arranged by considering algebraic multiplicity and with decreasing modulus, where $v(A)$ $(\leq \infty)$ is a sum of algebraic multiplicities of all nonzero eigenvalues of A. If $A \in \sigma_1$, then $\sum_{j=1}^{v(A)} \mu_j(A)$ is called the trace of A and is denoted by $\operatorname{sp} A$.

LIVŠIC'S THEOREM [3, p. 226]. *Let A be compact dissipative operator and let $A_J \in \sigma_1$. In order that the system of all root vectors of A be complete, it is necessary and sufficient that*

$$\sum_{j=1}^{v(A)} \operatorname{Im} \mu_j(A) = \operatorname{sp} A_J.$$

The determinant

$$\det(I - \mu A) = \prod_{j=1}^{v(A)} \left[1 - \mu\mu_j(A)\right], \qquad A \in \sigma_1,$$

is called the characteristic determinant of A and is denoted by $D_A(\mu)$. The characteristic determinant $D_A(\mu)$ is an entire function of μ, since for any $A \in \sigma_1$,

$$\sum_{j=1}^{v(A)} \left|\mu_j(A)\right| < \infty.$$

For any $A \in \sigma_2$, the regularized characteristic determinant is defined by

$$\tilde{D}_A(\mu) = \sum_{j=1}^{v(A)} \left[1 - \mu\mu_j(A)\right] e^{\mu\mu_j(A)}. \tag{2.1}$$

If the operator $I - \mu A$ has a bounded inverse defined on the whole space H, then the complex number μ is called an F-regular point (regular in the sense of Fredholm) for A.

Let A and B be linear bounded operators in H and $A - B \in \sigma_1$. If the point μ is an F-regular point of B, then

$$(I - \mu A)(I - \mu B)^{-1} = I - \mu(A - B)(I - \mu B)^{-1},$$

where $\mu(A - B)(I - \mu B)^{-1} \in \sigma_1$. Consequently, the determinant

$$D_{A/B}(\mu) = \det\left[(I - \mu A)(I - \mu B)^{-1}\right]$$

is meaningful and is called the determinant of perturbation of the operator B by the operator $K = A - B$.

We will use the following two well-known theorems.

THEOREM 2.1 [3, p. 172]. *If $A, B \in \sigma_2$, $A - B \in \sigma_1$, and μ is an F-regular point of B, then*

$$D_{A/B}(\mu) = \frac{\tilde{D}_A(\mu)}{\tilde{D}_B(\mu)} e^{\mu\, \mathrm{sp}(B-A)}.$$

THEOREM 2.2 [3, p. 177]. *Let A and B be bounded dissipative operators (in particular, one of them or both may be self-adjoint) and $A - B \in \sigma_1$. Then for any β_0 $(0 < \beta_0 < \frac{\pi}{2})$ the relation*

$$\lim_{\rho \to \infty} \left[\frac{1}{\rho} \log\left| D_{A/B}(\rho e^{i\beta}) \right| \right] = 0$$

holds uniformly with respect to β in the sector

$$\left\{ \lambda : \lambda = \rho e^{i\beta}, 0 < \rho < \infty, \left| \frac{\pi}{2} - \beta \right| < \beta_0 \right\}.$$

Let g be an entire function. If for each $\varepsilon > 0$ there exists a finite constant $C_\varepsilon > 0$, such that

$$|g(\lambda)| \le C_\varepsilon e^{\varepsilon|\lambda|}, \qquad \lambda \in \mathbf{C}, \tag{2.2}$$

then g is called an entire function of order ≤ 1 of growth and minimal type. It is evident from (2.2) that

$$\limsup_{|\lambda| \to \infty} \frac{1}{|\lambda|} \log|g(\lambda)| \le 0. \tag{2.3}$$

It is known that each function g, having property (2.2) and $g(0) = -1$, has the representation

$$g(\lambda) = -\lim_{r\to\infty} \prod_{|\lambda_j|\le r} \left(1 - \frac{\lambda}{\lambda_j}\right), \tag{2.4}$$

and also the limit $\lim_{r\to\infty}\Sigma_{|\lambda_j|\le r} 1/\lambda_j$ exists and is finite [6].

3. COMPLETENESS OF THE SYSTEM OF ROOT VECTORS OF L AND M

LEMMA 3.1. *Zero is not an eigenvalue of* L; *i.e.*, $\ker L = \{0\}$.

Proof. Let $y \in D(L)$ and $Ly = 0$. Then

$$-(p(x)y')' + q(x)y = 0, \tag{3.1}$$

and the function y satisfies the boundary condition (1.3). Therefore, there exists the constants c_1 and c_2, such that $y(x) = c_1 z(x) + c_2 u(x)$. Substituting this in the boundary conditions (1.3), we find $c_1 = c_2 = 0$; consequently $y = 0$. ∎

From Lemma 3.1 we get that there exists the inverse operator L^{-1}. Let us consider the functions $u(x)$ and $v(x) = z(x) - hu(x)$. These functions belong to $L_2(\mathbf{R}_+)$. The first satisfies the boundary condition at zero in (1.3) and the second at infinity. The functions $u(x)$ and $v(x)$ form a fundamental system of solutions of (3.1).

Let K denote the integral operator defined by the formula

$$Kf = \int_0^\infty G(x,t)f(t)\,dt, \qquad f \in L_2(\mathbf{R}_+), \tag{3.2}$$

where

$$G(x,t) = \begin{cases} u(x)v(t), & 0 \le x \le t, \\ u(t)v(x), & t \le x < \infty. \end{cases} \tag{3.3}$$

Since

$$\int_0^\infty \int_0^\infty |G(x,t)|^2\,dx\,dt < \infty,$$

we get that $K \in \sigma_2$. It is easy to verify that $K = L^{-1}$. Consequently, the root lineals of the operators L and K coincide and, therefore, the completeness in $L_2(\mathbf{R}_+)$ of the system of all eigenvectors and associated vectors of L is equivalent to the completeness of those for K. Since the algebraic multiplicity of nonzero eigenvalues of a compact operator is finite, each eigenvector of L may have only a finite number of linear independent associated vectors.

Let

$$a_1(\lambda) := [\varphi(x,\lambda), z(x)]_\infty, \qquad a_2(\lambda) := [\varphi(x,\lambda), u(x)]_\infty,$$

where $\varphi(x, \lambda)$ the solution of (1.2) given in the introduction.

It is evident that

$$\sigma_d(L) = \{\lambda : \lambda \in \mathbf{C}, a(\lambda) = 0\},$$

where $\sigma_d(L)$ denotes the set of all eigenvalues of L and

$$a(\lambda) = a_1(\lambda) - ha_2(\lambda). \tag{3.4}$$

THEOREM 3.2. *The functions* $a_j(\lambda)$, $j = 1, 2$, *are entire functions of order* ≤ 1 *of growth and minimal type.*

Proof. We set

$$a_{b,1}(\lambda) := [\varphi(x,\lambda), z(x)]_b = p(b)[\varphi(b,\lambda)z'(b) - \varphi'(b,\lambda)z(b)],$$
$$a_{b,2}(\lambda) := [\varphi(x,\lambda), u(x)]_b = p(b)[\varphi(b,\lambda)u'(b) - \varphi'(b,\lambda)u(b)].$$

Since for arbitrary fixed b, the functions $\varphi(b, \lambda)$ and $\varphi'(b, \lambda)$ are entire functions of λ of order $\frac{1}{2}$, consequently, the functions $a_{b,j}(\lambda)$, $j = 1, 2$, have the same property. Now we prove that the entire function $a_{b,j}(\lambda)$ converges to $a_j(\lambda)$ as $b \to \infty$, uniformly in λ in each compact set of the complex plane $\mathbf{C}$.

Let $y = y(x, \lambda)$ be a solution of (1.2); then

$$y = [y, u]_x z - [y, x]_x u. \tag{3.5}$$

If we define

$$f_1(x,\lambda) = [y, z]_x, \qquad f_2(x,\lambda) = [y, u]_x,$$

then following [2], we get that $f_1(x, \lambda)$ and $f_2(x, \lambda)$ satisfy a system of first order differential equations

$$\frac{\partial}{\partial x} f_1(x,\lambda) = \lambda y(x,\lambda)z(x), \quad \frac{\partial}{\partial x} f_2(x,\lambda) = \lambda y(x,\lambda)u(x), \qquad x \in \mathbf{R}_+.$$

Using (3.5) we obtain

$$\frac{\partial}{\partial x} f(x,\lambda) = \lambda Q(x) f(x,\lambda), \qquad x \in \mathbf{R}_+, \tag{3.6}$$

where

$$f(x,\lambda) = \begin{pmatrix} f_1(x,\lambda) \\ f_2(x,\lambda) \end{pmatrix}, \qquad Q(x) = \begin{pmatrix} -z(x)u(x) & z^2(x) \\ -u^2(x) & z(x)u(x) \end{pmatrix},$$

and the elements of $Q(x)$ are in $L_1(\mathbf{R}_+)$. For

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

we put $\|w\| = |w_1| + |w_2|$ and the norm of a square 2×2 matrix will be denoted by $\|.\|$. The inclusion $\|Q(x)\| \in L_1(\mathbf{R}_+)$ holds.

If $y(x,\lambda) = \varphi(x,\lambda)$, then the system (3.6) is equivalent to the integral equation

$$f(x,\lambda) = f(b,\lambda) + \lambda \int_b^x Q(t) f(t,\lambda)\, dt, \qquad x \in \mathbf{R}_+, \tag{3.7}$$

where

$$f(b,\lambda) = \begin{pmatrix} a_{b,1}(\lambda) \\ a_{b,2}(\lambda) \end{pmatrix}, \qquad f(0,\lambda) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad f(\infty,\lambda) = \begin{pmatrix} a_1(\lambda) \\ a_2(\lambda) \end{pmatrix}.$$

From (3.7) we find that

$$\| f(x,\lambda) \| \leq \| f(b,\lambda) \| \exp\left(|\lambda| \int_b^x \| Q(t) \|\, dt \right);$$

hence

$$\| f(\infty,\lambda) - f(b,\lambda) \| \leq |\lambda| \left(\int_b^\infty \| Q(t) \|\, dt \right) \exp\left(|\lambda| \int_0^\infty \| Q(t) \|\, dt \right), \tag{3.8}$$

$$\| f(\infty,\lambda) \| \leq \| f(b,\lambda) \| \exp\left(|\lambda| \int_b^\infty \| Q(t) \|\, dt \right). \tag{3.9}$$

It follows from (3.8) that $a_{bj}(\lambda)$ converges to $a_j(\lambda)$ as $b \to \infty$, uniformly in λ in a compact set. Consequently $a_j(\lambda)$, $j = 1, 2$, are entire functions.

For $b = 0$, from (3.9) we get

$$\| f(\infty, \lambda) \| \leq \exp\left(|\lambda| \int_b^\infty \| Q(t) \| \, dt \right);$$

hence $a_j(\lambda)$ are of not higher than the first order. Since, for arbitrary fixed b, the functions $a_{b,j}(\lambda)$, $j = 1, 2$, are entire functions of λ of order $\frac{1}{2}$, from (3.9) we obtain that the entire functions $a_j(\lambda)$, $j = 1, 2$, are of minimal type. ∎

Using Green's formula (1.1) as $b \to \infty$, we have

$$a_1(\lambda) = [\varphi(x, \lambda), z(x)]_\infty = -1 + \lambda \int_0^\infty \varphi(x, \lambda) z(x) \, dx, \quad (3.10)$$

$$a_2(\lambda) = [\varphi(x, \lambda), u(x)]_\infty = \lambda \int_0^\infty \varphi(x, \lambda) u(x) \, dx. \quad (3.11)$$

From (3.4), (3.10), and (3.11) we find that $a(0) = -1$.

We will use the well-known formula

$$[y_1, y_2]_x = \det \begin{pmatrix} [y_1, u]_x, [y_1, z]_x \\ [u, y_2]_x, [z, y_2]_x \end{pmatrix}, \qquad 0 \leq x \leq \infty, \quad (3.12)$$

where $y_1, y_2 \in \Omega$ [4, 5].

THEOREM 3.3. *The operator L is dissipative.*

Proof. Let $y \in D(L)$; then by Green's formula (1.1) we get

$$(Ly, y) - (y, Ly) = [y, y]_\infty - [y, y]_0. \quad (3.13)$$

From the boundary condition (1.3) and (3.12) we obtain that

$$[y, y]_0 = 0, \qquad [y, y]_\infty = 2i(\operatorname{Im} h) |[y, u]_\infty|^2,$$

or by (3.13)

$$\operatorname{Im}(Ly, y) = (\operatorname{Im} h) |[y, u]_\infty|^2 \geq 0. \quad (3.14)$$

Consequently, the operator L is dissipative. ∎

It follows from Theorem 3.3 that all the eigenvalues of L lie in the closed upper half-plane $\operatorname{Im} \lambda \geq 0$.

THEOREM 3.4. *The operator L has no real eigenvalue.*

Proof. Let λ_0 be a real eigenvalue of L and let the function $\varphi_0(x) = \varphi(x, \lambda_0)$ be the corresponding eigenfunction. Since $\operatorname{Im}(L\varphi_0, \varphi_0) = \operatorname{Im}(\lambda_0\|\varphi_0\|^2) = 0$, from (3.14) we find that $[\varphi_0, u]_\infty = 0$. By the boundary condition (1.3) we have $[\varphi_0, z]_\infty = 0$. Let $\theta_0(x) = \theta(x, \lambda_0)$; then using (3.12) we get

$$\begin{aligned} 1 &= W_0[\theta_0, \varphi_0] = W_\infty[\theta_0, \varphi_0] = [\theta_0, \varphi_0]_\infty \\ &= [\theta_0, u]_\infty[z, \varphi_0]_\infty - [\theta_0, z]_\infty[u, \varphi_0]_\infty = 0, \end{aligned}$$

which is a contradiction whence the result. ∎

Since $v(x) = z(x) - hu(x)$, setting $h = h_1 + ih_2$ we get from (3.2) in view of (3.3) that $K = K_1 + iK_2$, where

$$K_1 f = \int_0^\infty G_1(x,t) f(t)\, dt, \qquad K_2 f = \int_0^\infty G_2(x,t) f(t)\, dt,$$

and

$$G_1(x,t) = \begin{cases} u(x)[z(t) - h_1 u(t)], & 0 \le x \le t, \\ u(t)[z(x) - h_1 u(x)], & t \le x \le \infty, \end{cases}$$

$$G_2(x,t) = -h_2 u(x) u(t), \qquad h_2 = \operatorname{Im} h > 0.$$

The operator K_1 is the self-adjoint Hilbert–Schmidt operator in $L_2(\mathbf{R}_+)$ and K_2 is the self-adjoint one-dimensional operator in $L_2(\mathbf{R}_+)$, and $(K_2 f, f) \le 0$ for all $f \in L_2(\mathbf{R}_+)$.

Let L_1 denote the operator generated in $L_2(\mathbf{R}_+)$ by the differential expression $\ell(y)$ and the boundary condition

$$y(0)\cos\alpha + p(0)y'(0)\sin\alpha = 0, \qquad [y, z]_\infty - h_1[y, u]_\infty = 0.$$

It is easy to verify that K_1 is the inverse of L_1: $K_1 = L_1^{-1}$.

Let $T = -K$ and $T = T_1 + iT_2$, where $T_1 = -K_1$, $T_2 = -K_2$.

We will denote by λ_j and γ_k the eigenvalues of the operators L and L_1, respectively. Then the eigenvalues of T are $(-1/\lambda_j)$ and the eigenvalues of T_1 are $(-1/\gamma_k)$. Since L_1 is a self-adjoint operator, therefore $\operatorname{Im}\gamma_k = 0$ for all k.

THEOREM 3.5. $\sum_j \operatorname{Im}(-1/\lambda_j) = \operatorname{sp} T_2$.

Proof. Using Theorem 2.1 for $A = T_1$ and $B = T$ we obtain

$$D_{T_1/T}(\mu) = \frac{\tilde{D}_{T_1}(\mu)}{\tilde{D}_T(\mu)} e^{i\mu \operatorname{sp} T_2}, \tag{3.15}$$

and by (2.1) we get

$$\tilde{D}_T(\mu) = \prod_j \left(1 + \frac{\mu}{\lambda_j}\right) e^{-\mu/\lambda_j}, \qquad \tilde{D}_{T_1}(\mu) = \prod_k \left(1 + \frac{\mu}{\gamma_k}\right) e^{-\mu/\gamma_k}.$$

We set

$$a(\mu) = a_1(\mu) - h a_2(\mu), \qquad d(\mu) = a_1(\mu) - h_1 a_2(\mu),$$

where the functions $a_1(\mu)$ and $a_2(\mu)$ are given by (3.10) and (3.11). The eigenvalues of K and K_1 coincide with the root of the functions $a(\mu)$ and $d(\mu)$, respectively. Theorem 3.2 shows that the functions $a(\mu)$ and $d(\mu)$ are entire functions of order ≤ 1 of growth and minimal type and $a(0) = d(0) = -1$; therefore

$$a(\mu) = -\prod_j \left(1 + \frac{\mu}{\lambda_j}\right), \qquad d(\mu) = -\prod_k \left(1 - \frac{\mu}{\gamma_k}\right),$$

by (2.4). So

$$\tilde{D}_T(\mu) = -a(-\mu) e^{-\mu \Sigma_j (1/\lambda_j)}, \qquad \tilde{D}_{T_1}(\mu) = -d(-\mu) e^{-\mu \Sigma_k (1/\gamma_k)},$$

and from (3.15) we find

$$D_{T_1/T}(\mu) = \frac{d(-\mu)}{a(-\mu)} \exp\left(\mu \sum_j \frac{1}{\lambda_j} - \mu \sum_k \frac{1}{\gamma_k} + i\mu \operatorname{sp} T_2\right).$$

If we take $\mu = it$ $(0 < t < \infty)$, then we get

$$\frac{1}{t} \log\left|D_{T_1/T}(it)\right| = \frac{1}{t} \log|d(-it)| - \frac{1}{t} \log|a(-it)| - \sum_j \operatorname{Im} \frac{1}{\lambda_j} - \operatorname{sp} T_2. \tag{3.16}$$

From Theorem 2.2 and (2.3) we obtain that

$$\lim_{t \to \infty} \frac{1}{t} \log\left|D_{T_1/T}(it)\right| = 0, \tag{3.17}$$

and

$$\limsup_{t \to \infty} \frac{1}{t} \log|a(-it)| \leq 0, \qquad \limsup_{t \to \infty} \frac{1}{t} \log|d(-it)| \leq 0. \tag{3.18}$$

On the other hand, for $t > 0$,

$$\left|1 + \frac{it}{\lambda_j}\right|^2 \geq 1, \qquad \left|1 + \frac{it}{\gamma_k}\right|^2 \geq 1,$$

and we have $|a(-it)| \geq 1$, $|d(-it)| \geq 1$ for all $t > 0$. Consequently,

$$\frac{1}{t} \log|a(-it)| \geq 0, \qquad \frac{1}{t} \log|d(-it)| \geq 0,$$

and from (3.18) it follows that

$$\lim_{t\to\infty} \frac{1}{t} \log|a(-it)| = \lim_{t\to\infty} \frac{1}{t} \log|d(-it)| = 0. \tag{3.19}$$

Hence we get, by (3.16), (3.17), and (3.19) that

$$\sum_j \operatorname{Im}\left(-\frac{1}{\lambda_j}\right) = \operatorname{sp} T_2.$$

∎

Thus the operator T carries out all the conditions of Livšic theorem on completeness. Hence we have the following.

COROLLARY 3.6. *The system of all root vectors of the dissipative operator T (also of K) is complete in $L_2(\mathbf{R}_+)$.*

Since the completeness in $L_2(\mathbf{R}_+)$ of the system of all eigenvectors and associated vectors of L is equivalent to the completeness of those for K, from Corollary 3.6 we get

THEOREM 3.7. *The system of all eigenvectors and associated vectors of L is complete in $L_2(\mathbf{R}_+)$.*

Analogously for the operator M we obtain

THEOREM 3.8. *The system of all root vectors of the dissipative operator M is complete in $L_2(\mathbf{R}_+)$.*

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